ON MINIMAL FREE RESOLUTIONS OF SUB-PERMANENTS AND OTHER IDEALS ARISING IN COMPLEXITY THEORY

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Abstract. The minimal free resolution of the Jacobian ideals of the determinant polynomial were computed by Lascoux [12], and it is an active area of research to understand the Jacobian ideals of the permanent, see e.g., [13, 9]. As a step in this direction we compute several new cases and completely determine the linear strands of the minimal free resolutions of the ideals generated by sub-permanents.

Our motivation is to lay groundwork for the use of commutative algebra in algebraic complexity theory, building on the use of Hilbert functions in [8]. We also compute several Hilbert functions relevant for complexity theory.

1. Introduction

We study homological properties of *permanental ideals*: ideals generated by the $\kappa \times \kappa$ sub-permanents of a generic $n \times n$ matrix. We focus on the Hilbert series and minimal free resolution of such ideals. It turns out that there is a close connection to determinantal ideals, as well as to ideals generated by the set of square-free monomials in n variables. Our approach uses commutative algebra, combinatorics, and representation theory.

Our motivation comes from complexity theory: we hope to lay groundwork for generalizations of the *method of shifted partial derivatives* [8] via commutative algebra. In a companion paper [5], we explore the utility and limits of this method, and in future work we plan to prove new lower complexity bounds by combining the results of this paper and [5]. See [10] for a description of the method of shifted partial derivatives in geometric language.

Let $V = \mathbb{C}^N$, let $S^nV = \mathbb{C}[x_1, \cdots, x_N]_n$ denote the space of homogeneous polynomials of degree n on V^*, and let $Sym(S^nV)^*$ denote the space of all polynomials on S^nV.

The permanent polynomial is

$$\mathrm{perm}_m(y) = \sum_{\sigma \in \mathfrak{S}_m} y^1_{\sigma(1)} \cdots y^m_{\sigma(m)} \in S^m\mathbb{C}^{m^2},$$

where $y = (y^i_j)$, $1 \le i, j \le m$, are coordinates on $\mathbb{C}^{m^2}$ and $\mathfrak{S}_m$ denotes the group of permutations on m elements.

Write $\det_n(x) \in S^n\mathbb{C}^{n^2}$ for the determinant polynomial.

1.1. Hilbert functions and minimal free resolutions.

For an ideal $\mathcal{I} \subset Sym(V)$, the function $t \mapsto \dim \mathcal{I}_t$ is called the *Hilbert function* of $\mathcal{I}$. The method of shifted partial derivatives is a comparison of the Hilbert functions of the $(n-\kappa)$-th Jacobian ideals of two polynomials. For complexity theory, the most important polynomials are the permanent and the determinant. There is a substantial literature computing Hilbert functions of ideals, and more generally their

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minimal free resolutions. A minimal free resolution of $\mathcal{I}$ is an exact sequence of free $Sym(V)$-modules

$$0 \to F_q \to F_{q-1} \to \cdots \to F_0 \to \mathcal{I} \to 0, \tag{1}$$

with image$(F_i) \subseteq \mathfrak{m}F_{i-1}$ where $\mathfrak{m}$ denotes the maximal ideal in $Sym(V)$ generated by the linear forms V. Each $F_j = Sym(V) \cdot M_j$ for some graded vector space M_j, which may be taken to be a G-module if $\mathcal{I}$ is invariant under $G \subset GL(V)$. Then $\dim \mathcal{I}_t = \sum_{j=0}^{q} (-1)^j \dim F_{j,t}$. The module M_1 is the space of generators of $\mathcal{I}$ and the module M_2 is called the space of *syzygies* of $\mathcal{I}$. Especially important for our study is Lascoux's computation of the minimal free resolution of $\mathcal{I}^{\det_{n,\kappa}}$. In the case of permanental ideals, very little is known: in [13], Laubenbacher and Swanson determine a Gröbner basis for the 2×2 sub-permanents, as well as the radical and primary decomposition of this ideal. In the case of 3×3 sub-permanents, Kirkup [9] describes the structure of the minimal primes. Interestingly, the motivation for the work comes from the Alon-Jaeger-Tarsi conjecture on matrices over a finite field [2].

Recall that $\mathfrak{S}_n$ denotes the permutation group on n elements. It acts on basis vectors of $\mathbb{C}^n$ (the Weyl group action), and if we write $E = \mathbb{C}^n$, we will write $\mathfrak{S}_E$ to denote this action. We take $E, F = \mathbb{C}^n$. For example, $\mathrm{perm}_n \in S^n(E \otimes F)$ is acted on trivially by $\mathfrak{S}_E \times \mathfrak{S}_F$, so the generating modules in the ideal of the minimal free resolution of its Jacobian ideals will be $\mathfrak{S}_E \times \mathfrak{S}_F$-modules. If $H \subset G$ is a subgroup of a finite group G, and W an H-module, let $\mathbb{C}[G]$ denote the group algebra of G and let $Ind_H^G W := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ denote the induced G-module, see, e.g. [6, §3.3] for details. Irreducible representations of $\mathfrak{S}_n$ are indexed by partitions of n. If π is such a partition, we let $[\pi]$ denote the corresponding $\mathfrak{S}_n$-module. If $\pi = (a, 1, \cdots, 1)$ with b 1's, we write $\pi = (a, 1^b)$. For the next result, give $E = \mathbb{C}^n$ a basis $e_1, \cdots, e_n$ and let $E_q = \mathrm{span}\{e_1, \cdots, e_q\}$, and similarly for $F = \mathbb{C}^n$.

Theorem 1.1. *Let M_j denote the module generating the j-th term of the minimal free resolution of $\mathcal{I}^{\mathrm{perm}_n,\kappa}$, the ideal generated by size κ-sub-permanents of an $n \times n$ matrix with variable entries, and let $M_{j,\kappa+j-1}$ denote its linear component. Then* $\dim M_{j,\kappa+j-1} = \binom{n}{\kappa+j-1}^2 \binom{2(\kappa+j-2)}{j-1}$. *As an $\mathfrak{S}_E \times \mathfrak{S}_F$-module,*

$$M_{j,\kappa+j-1} = Ind^{\mathfrak{S}_E \times \mathfrak{S}_F}_{(\mathfrak{S}_{E_{\kappa+j-1}} \times \mathfrak{S}_{n-(\kappa+j-1)}) \times (\mathfrak{S}_{F_{\kappa+j-1}} \times \mathfrak{S}_{n-(\kappa+j-1)})} \Big(\bigoplus_{a+b=j-1} ([\kappa+b, 1^a]_{E_{\kappa+j-1}} \otimes [n-(\kappa+j-1)]) \otimes ([\kappa+a, 1^b]_{F_{\kappa+j-1}} \otimes [n-(\kappa+j-1)]) \Big). \tag{2}$$

Compare Theorem 1.1 with Theorem 1.3 below.

Theorem 1.2. *Let $\mathcal{I}_t^{\mathrm{perm}_n,2}$ denote the degree t component of the ideal generated by the size two sub-permanents of an $n \times n$ matrix. Then* $\dim \mathcal{I}_2^{\mathrm{perm}_n,2} = \binom{n}{2}^2$. *For $3 \le t \le n$:*

$$\dim \mathcal{I}_t^{perm_n,2} = \binom{n^2+t-1}{t} - \Big[\binom{n}{t}^2 + n^2 + (t-1)\Big(\binom{n^2}{2} - \binom{n}{2}^2\Big) + 2\binom{t-1}{2}\Big(\binom{n}{2}^2 + n\binom{n}{3}\Big) + 2n \sum_{j=3}^{t-1} \binom{t-1}{j}\binom{n}{j+1}\Big],$$

and for $t > n$:

$$\dim \mathcal{I}_t^{perm_n,2} = \binom{n^2+t-1}{t} - \Big[n^2 + (t-1)\Big(\binom{n^2}{2} - \binom{n}{2}^2\Big) + 2\binom{t-1}{2}\Big(\binom{n}{2}^2 + n\binom{n}{3}\Big) + 2n \sum_{j=3}^{t-1} \binom{t-1}{j}\binom{n}{j+1}\Big].$$

The latter formula is $\dim S^t\mathbb{C}^{n^2}$ *minus the value of the Hilbert polynomial of* $\mathcal{I}^{\mathrm{perm}_n,2}$ *at* t.

Information from the minimal free resolution might lead to more modules of polynomials that one could use in complexity theory beyond the shifted partial derivatives.

1.2. **Jacobian ideals of $x_1\cdots x_n$.** Another polynomial that arises in complexity theory is the monomial $x_1\cdots x_n$. Note that $\mathrm{rank}(x_1\cdots x_n)_{n-\kappa,\kappa}=\binom{n}{\kappa}$ and $\mathrm{rank}(\mathrm{perm}_n)_{n-\kappa,\kappa}=\binom{n}{\kappa}^2$.

Theorem 1.3. *Let* $\mathcal{I}^{x_1\cdots x_n,\kappa}$ *denote the ideal generated by the derivatives of order* $n-\kappa$ *of the polynomial* $x_1\cdots x_n$*, i.e., the ideal generated by the set of square free monomials of degree* κ *in* n *variables. The associated coordinate ring* $Sym(V)/\mathcal{I}^{x_1\cdots x_n,\kappa}$ *is Cohen-Macaulay and its minimal free resolution is linear. As an* $\mathfrak{S}_n$*-module, the generators of the* j*-th term in the minimal free resolution of* $\mathcal{I}^{x_1\cdots x_n,\kappa}$ *is*

$$M_j=M_{j,\kappa+j-1}=Ind_{\mathfrak{S}_{\kappa+j}}^{\mathfrak{S}_n}[\kappa,1^{j-1}],$$

which has dimension $\binom{\kappa+j-1}{j}\binom{n}{\kappa+j}$.

Remark 1.4. Theorem 1.3 overlaps with the results of [3], as the ideals generated by square-free monomials are a special case of the DeConcini-Procesi ideals of hooks discussed in [3], but Theorem 1.3 gives more precise information for this special case.

1.3. **Young flattenings.** The method of shifted partial derivatives fits into a general theory of *Young flattenings* developed in [11], which is a method for finding determinantal equations on spaces of polynomials invariant under a group action. The motivation in [11] was to obtain lower bounds for symmetric tensor border rank, that is for the expression of a polynomial as a sum of n-th powers of linear forms. In future work we plan to explore the extent that Young flattenings can prove circuit lower bounds. We hope to do this via information extracted from minimal free resolutions of Jacobian ideals, e.g. by tensoring such with a Koszul sequence and taking Hilbert functions.

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2. The minimal free resolution of the ideal generated by minors of size κ

This section, except for §2.3, is expository. The results are due to Lascoux [12]. The results in §2.3 were known in slightly different language, but to our knowledge are only available in an unpublished manuscript of Roberts [17]. For the other subsections, we follow the presentation in [19].

2.1. **Statement of the result.** Let $E,F=\mathbb{C}^n$, give $E\otimes F$ coordinates (x^i_j), with $1\le i,j\le n$. Set $r=\kappa-1$. Let $\hat{\sigma}_r=\hat{\sigma}_r(Seg(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}))\subset\mathbb{C}^n\otimes\mathbb{C}^n=E^*\otimes F^*$ denote the variety of $n\times n$ matrices of rank at most r. By "degree $S_\pi E$", we mean $|\pi|=p_1+\cdots+p_n$. Write $\ell(\pi)$ for the largest j such that $p_j>0$. Write $\pi+\pi'=(p_1+p_1',\cdots,p_n+p_n')$.

The *weight* (under $GL(E)\times GL(F)$) of a monomial $x^{i_1}_{j_1}\cdots x^{i_q}_{j_q}\in S^q(E\otimes F)$ is given by a pair of n-tuples $((w^E_1,\cdots,w^E_n),(w^F_1,\cdots,w^F_n))$ where w^E_s is the number of i_α's equal to s and w^F_t is the number of j_α's equal to t. A vector is a weight vector of weight $((w^E_1,\cdots,w^E_n),(w^F_1,\cdots,w^F_n))$ if it can be written as a sum of monomials of weight $((w^E_1,\cdots,w^E_n),(w^F_1,\cdots,w^F_n))$. Any $GL(E)\times GL(F)$-module has a basis of weight vectors, and any irreducible module has a unique *highest weight* which (if the representation is polynomial) is a pair of partitions, $(\pi,\mu)=((p_1,\cdots,p_n),(m_1,\cdots,m_n))$,

where we allow a string of zeros to be added to a partition to make it of length n. The corresponding $GL(E)\times GL(F)$-module is denoted $S_\pi E\otimes S_\mu F$.

Theorem 2.1. [12] *Let $0\to F_N\to\cdots\to F_1\to Sym(E\otimes F)=F_0\to\mathbb{C}[\hat\sigma_r]\to 0$ denote the minimal free resolution of $\hat\sigma_r$. Then*

(1) *$N=(n-r)^2$, i.e., $\hat\sigma_r$ is arithmetically Cohen-Macaulay.*
(2) *$\hat\sigma_r$ is Gorenstein, i.e., $F_N=Sym(E\otimes F)$, generated by $S_{(n-r)^n}E\otimes S_{(n-r)^n}F$. In particular $F_{N-j}\simeq F_j$ as $SL(E)\times SL(F)$- modules, although they are not isomorphic as $GL(E)\times GL(F)$-modules.*
(3) *For $1\le j\le N-1$, the space F_j has generating modules of degree $sr+j$ where $1\le s\le\lfloor\sqrt{j}\rfloor$. The modules of degree $r+j$ form the generators of the linear strand of the minimal free resolution.*
(4) *The generating module of F_j is multiplicity free.*
(5) *Let α,β be (possibly zero) partitions such that $\ell(\alpha),\ell(\beta)\le s$. Independent of the lengths (even if they are zero), write $\alpha=(\alpha_1,\cdots,\alpha_s)$, $\beta=(\beta_1,\cdots,\beta_s)$. The degree $sr+j$ generators of F_j, for $1\le j\le N$ are*

$$M_{j,rs+j}=\bigoplus_{s\ge1}\bigoplus_{\substack{|\alpha|+|\beta|=j-s^2\\ \ell(\alpha),\ell(\beta)\le s}}S_{(s)^{r+s}+(\alpha,0^r,\beta')}E\otimes S_{(s)^{r+s}+(\beta,0^r,\alpha')}F. \quad (3)$$

The Young diagrams of the modules are depicted in Figure 1 below.

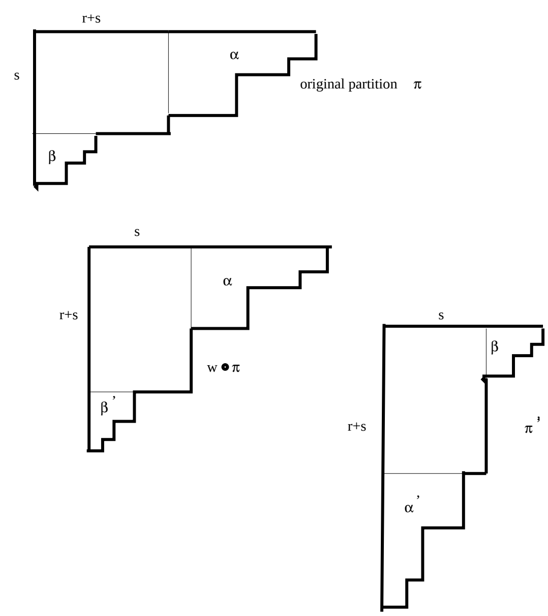

FIGURE 1. Partition π and pairs of partitions $(s)^{r+s}+(\alpha,0^r,\beta')=w\cdot\pi$ and $(s)^{r+s}+(\beta,0^r,\alpha')=\pi'$ it gives rise to in the resolution (see §2.4 for explanations).

(6) *In particular the generator of the linear component of F_j is*

$$M_{j,j+r}=\bigoplus_{a+b=j-1}=S_{a+1,1^{r+b}}E\otimes S_{b+1,1^{r+a}}F. \quad (4)$$

This module admits a basis as follows: form a size $r+j$ submatrix using $r+b+1$ distinct rows, repeating a subset of a rows to have the correct number of rows and $r+a+1$

distinct columns, repeating a subset of b columns, and then performing a "tensor Laplace expansion" as described below.

Remark 2.2. Our β is β' in [19].

2.2. **The Koszul resolution.** If $\mathcal{I} = Sym(V)$, the minimal free resolution is given by the exact complex

$$\cdots \to S^{q-1}V \otimes \Lambda^{p+2}V \to S^qV \otimes \Lambda^{p+1}V \to S^{q+1}V \otimes \Lambda^pV \to \cdots \tag{5}$$

The maps are given by the transpose of exterior derivative (Koszul) map $d_{p,q} : S^qV^* \otimes \Lambda^{p+1}V^* \to S^{q-1}V^* \otimes \Lambda^{p+2}V^*$. Write $d^T_{p,q} : S^{q-1}V \otimes \Lambda^{p+2}V \to S^qV \otimes \Lambda^{p+1}V$. We have the $GL(V)$-decomposition $S^qV \otimes \Lambda^{p+1}V = S_{q,1^{p+1}}V \oplus S_{q+1,1^p}V$, so the kernel of $d^T_{p,q}$ is the first module, which also is the image of $d^T_{p+1,q-1}$.

Explicitly, $d^T_{p,q}$ is the composition of polarization ($\Lambda^{p+2}V \to \Lambda^{p+1}V \otimes V$) and multiplication:

$$S^{q-1}V \otimes \Lambda^{p+2}V \to S^{q-1}V \otimes \Lambda^{p+1}V \otimes V \to S^qV \otimes \Lambda^{p+1}V.$$

For the minimal free resolution of any ideal, the linear strand will embed inside (5).

Throughout this article, we will view $S_{q+1,1^p}V$ as a submodule of $S^qV \otimes \Lambda^{p-1}V$, $GL(V)$-complementary to $d^T_{p,q}(S^{q-1,1^p}V)$.

For $T \in S^\kappa V \otimes V^{\otimes j}$, and $P \in S^\ell V$, introduce notation for multiplication on the first factor, $T \cdot P \in S^{\kappa+\ell}V \otimes V^{\otimes j}$. Write $F_j = M_j \cdot Sym(V)$. As always, $M_0 = \mathbb{C}$.

2.3. **Geometric interpretations of the terms in the linear strand** (4)**.** First note that $F_1 = M_1 \cdot Sym(E \otimes F)$, where $M_1 = M_{1,r+1} = \Lambda^{r+1}E \otimes \Lambda^{r+1}F$, the size $r+1$ minors which generate the ideal. The syzygies among these equations are generated by

$$M_{2,r+2} := S_{1^{r+2}}E \otimes S_{21^r}F \oplus S_{21^r}E \otimes S_{1^{r+2}}F \subset \mathcal{I}^{\sigma_r}_{r+2} \otimes V$$

(i.e., $F_2 = M_2 \cdot Sym(E \otimes F)$), where elements in the first module may be obtained by choosing $r+1$ rows and $r+2$ columns, forming a size $r+2$ square matrix by repeating one of the rows, then doing a 'tensor Laplace expansion" that we now describe:

In the case $r = 1$ we have highest weight vector

$$\begin{aligned} S^{1|12}_{123} &:= (x^1_2x^2_3 - x^2_2x^1_3) \otimes x^1_1 - (x^1_1x^2_3 - x^2_1x^1_3) \otimes x^1_2 + (x^1_1x^2_2 - x^1_2x^2_1) \otimes x^1_3 \\ &= M^{12}_{23} \otimes x^1_1 - M^{12}_{13} \otimes x^1_2 + M^{12}_{12} \otimes x^1_3 \end{aligned} \tag{6}$$

where in general M^I_J will denote the minor obtained from the submatrix with indices I, J. The expression (6) corresponds to the Young tableaux pair:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

To see (6) is indeed a highest weight vector, first observe that it has the correct weights in both E and F, and that in the F-indices $\{1,2,3\}$ it is skew and that in the first two E indices it is also skew. Finally to see it is a highest weight vector note that any raising operator sends it to zero. Also note that under the multiplication map $S^2V \otimes V \to S^3V$ the element maps to zero, because the map corresponds to converting a tensor Laplace expansion to an actual one, but the determinant of a matrix with a repeated row is zero.

In general, a basis of $S_\pi E \otimes S_\mu F$ is indexed by pairs of semi-standard Young tableau in π and μ. In the linear strand, all partitions appearing are hooks, a basis of $S_{a,1^b}E$ is given by two sequences of integers taken from $[n]$, one weakly increasing of length a and one strictly

increasing of length b, where the first integer in the first sequence is at least the first integer in the second sequence.

A highest weight vector in $S_{21^r}E\otimes S_{1^{r+2}}F$ is

$$S^{1|1,\cdots,r+1}_{1,\cdots,r+2} = M^{1,\cdots,r+1}_{2,\cdots,r+2}\otimes x^1_1 - M^{1,\cdots,r+1}_{1,3,\cdots,r+1}\otimes x^1_2 + \cdots + (-1)^r M^{1,\cdots,r+1}_{1,\cdots,r+1}\otimes x^1_{r+2},$$

and the same argument as above shows it has the desired properties. Other basis vectors are obtained by applying lowering operators to the highest weight vector, so their expressions will be more complicated.

Remark 2.3. If we chose a size $r+2$ submatrix, and perform a tensor Laplace expansion of its determinant about two different rows, the difference of the two expressions corresponds to a linear syzygy, but these are in the span of M_2. These expressions are important for comparison with the permanent, as they are the only linear syzygies for the ideal generated by the size $r+1$ sub-permanents, where one takes the permanental Laplace expansion.

Continuing, F_3 is generated by the module

$$M_{3,r+3} = S_{1^{r+3}}E\otimes S_{3,1^r}F \oplus S_{2,1^{r+1}}E\otimes S_{2,1^{r+1}}F \oplus S_{3,1^r}E\otimes S_{1^{r+3}}F \subset M_2\otimes V.$$

These modules admit bases of double tensor Laplace type expansions of a square submatrix of size $r+3$. In the first case, the highest weight vector is obtained from the submatrix whose rows are the first $r+3$ rows of the original matrix, and whose columns are the first r-columns with the first column repeated three times. For the second module, the highest weight vector is obtained from the submatrix whose rows and columns are the first $r+2$ such, with the first row/column repeated twice. A highest weight vector for $S_{3,1^r}E\otimes S_{1^{r+3}}F$ is

$$\begin{aligned} S^{11|1,\cdots,r+1}_{1,\cdots,r+3} &= \sum_{1\le\beta_1<\beta_2\le r+3}(-1)^{\beta_1+\beta_2}M^{1,\cdots,r+1}_{1,\cdots,\hat{\beta}_1,\cdots,\hat{\beta}_2,\cdots,r+3}\otimes(x^1_{\beta_1}\wedge x^1_{\beta_2}) \\ &= \sum_{\beta=1}^{r+3}(-1)^{\beta+1}S^{1|1,\cdots,i_{r+1}}_{1,\cdots,\hat{\beta},\cdots,r+3}\otimes x^1_\beta. \end{aligned}$$

Here $S^{1|1,\cdots,i_{r+1}}_{1,\cdots,\hat{\beta},\cdots,r+3}$ is defined in the same way as the highest weight vector.

A highest weight vector for $S_{2,1^{r+1}}E\otimes S_{2,1^{r+1}}F$ is

$$\begin{aligned} S^{1|1,\cdots,r+3}_{1|1,\cdots,r+2} &= \sum_{\alpha,\beta=1}^{r+3}(-1)^{\alpha+\beta}M^{1,\cdots,\hat{\alpha},\cdots,r+2}_{1,\cdots,\hat{\beta},\cdots,i+2}\otimes(x^\alpha_1\wedge x^1_\beta) \\ &= \sum_{\beta=1}^{r+3}(-1)^{\beta+1}S^{1,\cdots,r+2}_{1|1,\cdots,\hat{\beta},\cdots,r+2}\otimes x^1_\beta - \sum_{\alpha=1}^{r+3}(-1)^{\alpha+1}S^{1|1,\cdots,\hat{\alpha},\cdots,r+3}_{1,\cdots,r+2}\otimes x^\alpha_1. \end{aligned}$$

Here $S^{1,\cdots,r+2}_{1|1,\cdots,\hat{\beta},\cdots,r+2}$, $S^{1|1,\cdots,\hat{\alpha},\cdots,r+3}_{1,\cdots,r+2}$ are defined in the same way as the corresponding highest weight vectors.

Proposition 2.4. *The highest weight vector of* $S_{p+1,1^{r+q}}E\otimes S_{q+1,1^{r+p}}F \subset M_{p+q+1,r+p+q+1}$ *is*

$$S^{1^p|1,\cdots,r+q+1}_{1^q|1,\cdots,r+p+1} = \sum_{\substack{I\subset[r+q+1],|I|=q,\\ J\subset[r+p+1],|J|=p}}(-1)^{|I|+|J|}M^{1,\cdots,\hat{i}_1,\cdots,\hat{i}_q,\cdots,(r+q+1)}_{1,\cdots,\hat{j}_1,\cdots,\hat{j}_p,\cdots,(r+p+1)}\otimes(x^1_{j_1}\wedge\cdots\wedge x^1_{j_p}\wedge x^{i_1}_1\wedge\cdots\wedge x^{i_q}_1).$$

A hatted index is one that is omitted from the summation.

Proof. It is clear the expression has the correct weight and is a highest weight vector, and that it lies in $S^{r+1}V\otimes\Lambda^{p+q}V$. We now show it maps to zero under the differential.

Under the map $d^T : S^{r+1}V\otimes\Lambda^{p+q}V \to S^rV\otimes\Lambda^{p+q+1}V$, the element $S^{1^p|1,\cdots,r+q+1}_{1^q|1,\cdots,r+p+1}$ maps to:

$$\sum_{\substack{I\subset[r+q+1],|I|=q,\\ J\subset[r+p+1],|J|=p}}(-1)^{|I|+|J|}[\sum_{\alpha\in I}(-1)^{p+\alpha}M^{1,\cdots,\hat{i}_1,\cdots,\hat{i}_q,\cdots,(r+q+1)}_{1,\cdots,\hat{j}_1,\cdots,\hat{j}_p,\cdots,(r+p+1)}x_1^{i_\alpha}\otimes(x^1_{j_1}\wedge\cdots\wedge x^1_{j_p}\wedge x_1^{i_1}\wedge\cdots\wedge\hat{x}_1^{i_\alpha}\wedge\cdots\wedge x_1^{i_q})$$

$$+\sum_{\beta\in J}(-1)^{\beta}M^{1,\cdots,\hat{i}_1,\cdots,\hat{i}_q,\cdots,(r+q+1)}_{1,\cdots,\hat{j}_1,\cdots,\hat{j}_p,\cdots,(r+p+1)}x^1_{j_\beta}\otimes(x^1_{j_1}\wedge\cdots\wedge\hat{x}^1_{j_\beta}\wedge\cdots\wedge x^1_{j_p}\wedge x_1^{i_1}\wedge\cdots\wedge x_1^{i_q})]$$

Fix I and all indices in J but one, call the resulting index set J', and consider the resulting term

$$\sum_{\beta\in[r+p+1]\setminus J'}(-1)^{f(\beta,J')}M^{1,\cdots,\hat{i}_1,\cdots,\hat{i}_q,\cdots,(r+q+1)}_{1,\cdots,\hat{j}'_1,\cdots,\hat{j}'_{p-1},\cdots,(r+p+1)}x^1_\beta\otimes(x^1_{j'_1}\wedge\cdots\wedge x^1_{j'_{p-1}}\wedge x_1^{i_1}\wedge\cdots\wedge x_1^{i_q})$$

where $f(\beta,J')$ equals the number of $j'\in J$ less than β. This term is the Laplace expansion of the determinant of a matrix of size $r+1$ which has its first row appearing twice, and is thus zero. □

Notice that if $q,p>0$, then $S^{1^p|1,\cdots,r+q+1}_{1^q|1,\cdots,r+p+1}$ is the sum of terms including $S^{1^p|1,\cdots,r+q}_{1^{q-1}|1,\cdots,r+p+1}\otimes x_1^{r+q+1}$ and $S^{1^{p-1}|1,\cdots,r+q+1}_{1^q|1,\cdots,r+p}\otimes x^1_{r+p+1}$. This implies the following corollary:

Corollary 2.5 (Roberts [17])**.** *Each module $S_{a,1^{r+b}}E\otimes S_{b,1^{r+a}}F$, where $a+b=j$ that appears with multiplicity one in $F_{j,j+r}$, appears with multiplicity two in $F_{j-1,j+r}$ if $a,b>0$, and multiplicity one if a or b is zero. The map $F_{j,j+r+1}\to F_{j-1,j+r+1}$ restricted to $S_{a,1^{r+b}}E\otimes S_{b,1^{r+a}}F$, maps non-zero to both $(S_{a-1,1^{r+b}}E\otimes S_{b,1^{r+a-1}}F)\cdot E\otimes F$ and $(S_{a,1^{r+b-1}}E\otimes S_{b-1,1^{r+a}}F)\cdot E\otimes F$.*

Proof. The multiplicities and realizations come from applying the Pieri rule. (Note that if a is zero the first module does not exist and if b is zero the second module does not exist.) That the maps to each of these is non-zero follows from the remark above. □

Remark 2.6. In [17] it is proven more generally that all the natural realizations of the irreducible modules in M_j have non-zero maps onto every natural realization of the module in F_{j-1}. Moreover, the constants in all the maps are determined explicitly. The description of the maps is different than the one presented here.

2.4. **Proof of Theorem 2.1.** This section is expository and less elementary than the rest of the paper. The variety $\hat{\sigma}_r$ admits a desingularization by the geometric method of [19], namely consider the Grassmannian $G(r,E^*)$ and the vector bundle $p:\mathcal{S}\otimes F\to G(r,E^*)$ whose fiber over $x\in G(r,E^*)$ is $x\otimes F$. (Although we are breaking symmetry here, it will be restored in the end.) The total space admits the interpretation as the incidence variety

$$\{(x,\phi)\in G(r,E^*)\times\mathrm{Hom}(F,E^*)\mid\phi(F)\subseteq x\},$$

and the projection to $\mathrm{Hom}(F,E^*)=E^*\otimes F^*$ has image $\hat{\sigma}_r$. One also has the exact sequence

$$0\to\mathcal{S}\otimes F^*\to\underline{E^*\otimes F^*}\to Q\otimes F^*\to 0$$

where $\underline{E^*\otimes F^*}$ denotes the trivial bundle with fiber $E^*\otimes F^*$ and $Q=\underline{E^*}/\mathcal{S}$ is the quotient bundle. As explained in [19], letting $q:\mathcal{S}\otimes F^*\to E^*\otimes F^*$ denote the projection, q is a desingularization of $\hat{\sigma}_r$, the higher direct images $\mathcal{R}_iq^*(\mathcal{O}_{\mathcal{S}\otimes F^*})$ are zero for $i>0$, and so by [19, Thm. 5.12,5.13] one concludes $F_i=M_i\cdot Sym(E\otimes F)$ where

$$\begin{aligned}M_i&=\oplus_{j\geq0}H^j(G(r,E^*),\Lambda^{i+j}(\mathcal{Q}^*\otimes F))\\&=\oplus_{j\geq0}\oplus_{|\pi|=i+j}H^j(G(r,E^*),S_\pi Q)\otimes S_{\pi'}F\end{aligned}$$

One now uses the Bott-Borel-Weil theorem to compute these cohomology groups. An algorithm for this is given in [19, Rem. 4.1.5]: If $\pi = (p_1, \cdots, p_q)$ (where we must have $p_1 \leq n$ to have $S_{\pi'}F$ non-zero, and $q \leq n-r$ as rank$Q = n-r$), then $S_\pi Q^*$ is the vector bundle corresponding to the sequence

$$(0^r, p_1, \cdots, p_{n-r}). \tag{7}$$

The dotted Weyl action by $\sigma_i = (i, i+1) \in \mathfrak{S}_n$ is

$$\sigma_i \cdot (\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, \cdots, \alpha_n)$$

and one applies simple reflections to try to transform α to a partition until one either gets a partition after u simple reflections, in which case H^u is equal to the module associated to the partition one ends up with and all other cohomology groups are zero, or one ends up on a wall of the Weyl chamber, i.e., at one step one has $(\beta_1, \cdots, \beta_n)$ with some $\beta_{i+1} = \beta_i + 1$, in which case there is no cohomology.

In our case, we need to move p_1 over to the first position in order to obtain a partition, which means we need $p_1 \geq r+1$, and then if $p_2 < 2$ we are done, otherwise we need to move it etc... The upshot is we can get cohomology only if there is an s such that $p_s \geq r+s$ and $p_{s+1} < s+1$, in which case we get

$$S_{(p_1 - r, \cdots, p_s - r, s^r, p_{s+1}, \cdots, p_{n-r})} E \otimes S_{\pi'} F$$

contributing to H^{rs}. Say we are in this situation, then write $(p_1 - r - s, \cdots, p_s - r - s) = \alpha$, $(p_{s+1}, \cdots, p_{n-r}) = \beta'$, so

$$(p_1 - r, \cdots, p_s - r, s^r, p_{s+1}, \cdots, p_{n-r}) = (s^{r+s}) + (\alpha, 0^r, \beta')$$

and moreover we may write

$$\pi' = (s^{r+s}) + (\beta, 0^r, \alpha')$$

proving Theorem 2.1. The case $s = 1$ gives the linear strand of the resolution.

3. The minimal free resolution of the ideal generated by the space of square free monomials

The space of $(n-\kappa)$-th shifted partial derivatives of the polynomial $x_1 \cdots x_n \in S^n\mathbb{C}^n$ is spanned by the set of square free monomials in $S^\kappa\mathbb{C}^n$ (also called the *vectors of regular weight*, see §4.1). While the ideal these generate has been well-studied, we were unable to find its minimal free resolution in the literature.

Proposition 3.1. *The Hilbert function of $\mathcal{I}^{x_1 \cdots x_n, \kappa}$ in degree $\kappa + t$ is*

$$\dim \mathcal{I}^{x_1 \cdots x_n, \kappa}_{\kappa+t} = \sum_{j=0}^{n-\kappa} \binom{n}{\kappa - j} \binom{\kappa + t - 1}{\kappa + j - 1} \tag{8}$$

Proof. The ideal in degree $d = t + \kappa$ has a basis of the distinct monomials of degree d containing at least κ distinct indices. When we divide such a basis vector by $x_1 \cdots x_n$ the denominator will have degree at most κ. For each $i \leq \kappa$, the space of possible numerators with a denominator of degree i that is fixed, has dimension $\dim S^{d-n+i}\mathbb{C}^{n-i}$, and there are $\binom{n}{i}$ possible denominators. Summing over i gives the result. □

For the Hilbert function of the coordinate ring, we have the following expression:

Proposition 3.2. *The Hilbert function of $Sym(\mathbb{C}^n)/\mathcal{I}^{x_1 \cdots x_n, \kappa}$ in degree t is*

$$\dim (Sym(\mathbb{C}^n)/\mathcal{I}^{x_1 \cdots x_n, \kappa})_t = \sum_{j=0}^{n-\kappa-2} \binom{n}{j+1} \binom{t-1}{j}, \tag{9}$$

if $t \geq n-\kappa-1$, *and* $\binom{n+t-1}{n-1}$ *if* $t < n-\kappa-1$.

The expression (9) will be a consequence of the results of the next section.

3.1. **The minimal free resolution.**

Definition 3.3. [20] A simplicial complex Δ on a vertex set V is a collection of subsets σ of V, such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. If $|\sigma| = i+1$ then σ is called an i–face. Let $f_i(\Delta)$ denote the number of i-faces of Δ, and define $\dim(\Delta) = \max\{i \mid f_i(\Delta) \neq 0\}$. If $\dim(\Delta) = n-1$, we define $f_\Delta(t) = \sum_{i=0}^{n} f_{i-1}t^{n-i}$. The ordered list of coefficients of $f_\Delta(t)$ is called the f-vector of Δ, and the coefficients of $h_\Delta(t) := f_\Delta(t-1)$ is called the h-vector of Δ. The Alexander dual $\Delta^\vee$ of Δ [16] is the simplicial complex

$$\Delta^\vee = \{\tau \mid \overline{\tau} \notin \Delta\}$$

where $\overline{\tau}$ denotes the complement $V \smallsetminus \tau$.

For example, if Δ is the one skeleton of a three simplex, then $f(\Delta) = (1,4,6)$ and $h(\Delta) = (1,2,3)$, and $\Delta^\vee$ consists of the four vertices and the empty face.

Definition 3.4. Let Δ be a simplicial complex on vertices $\{x_1,\dots,x_n\}$. The Stanley-Reisner ideal $\mathcal{I}_\Delta$ is

$$\mathcal{I}_\Delta = \langle x_{i_1}\cdots x_{i_j} \mid \{x_{i_1},\dots,x_{i_j}\} \text{ is not a face of } \Delta\rangle \subseteq \mathbb{C}[x_1,\dots x_n],$$

and the Stanley-Reisner ring is $\mathbb{C}[x_1,\dots x_n]/\mathcal{I}_\Delta$.

The Stanley-Reisner ideal $\mathcal{I}_{\Delta^\vee}$ of $\Delta^\vee$ is obtained by monomializing the primary decomposition of $\mathcal{I}_\Delta$: for each primary component in the primary decomposition (for a square free monomial ideal, these are just collections of variables), take the product of the terms in the component. So if

$$\mathcal{I}_\Delta = \bigcap_j \langle x_{i_{j_1}},\dots,x_{i_{j_\kappa}}\rangle,$$

then $x_{i_{j_1}}\cdots x_{i_{j_\kappa}}$ is a minimal generator of $\mathcal{I}_{\Delta^\vee}$, and all minimal generators arise this way. Of special interest to us is the ideal $\mathcal{I}_{\Delta(n,\kappa)}$ generated by all square-free monomials of degree κ in n variables; it is the Stanley-Reisner ideal of the $\kappa-2$ skeleton of an $n-1$ simplex.

Lemma 3.5. [3, Lem. 2.8] *The quotient $Sym(V)/\mathcal{I}_{\Delta(n,\kappa)}$ is Cohen-Macaulay and has a minimal free resolution which is linear.*

We include a proof along the lines of the above discussion.

Proof. The ideal $\mathcal{I}_{\Delta(n,n-\kappa)}$ is the Stanley Reisner ideal of the $n-\kappa-2$ skeleton of the $n-1$ simplex Δ_{n-1}. The primary decomposition of $\mathcal{I}_{\Delta(n,n-\kappa)}$ is [18, Thm. 5.3.3]

$$\mathcal{I}_{\Delta(n,n-\kappa)} = \bigcap_{1\leq j_1<\cdots<j_{\kappa+1}\leq n} (x_{j_1},\dots,x_{j_{\kappa+1}}).$$

Thus, the Alexander dual ideal satisfies

$$\mathcal{I}_{\Delta(n,n-\kappa)^\vee} = \mathcal{I}_{\Delta(n,\kappa)}.$$

It follows that the Alexander dual of $\mathcal{I}_{\Delta(n,n-\kappa)}$ is the Stanley-Reisner ideal of the $\kappa-2$ skeleton of Δ_{n-1}, which is $\mathcal{I}_{\Delta(n,\kappa)}$. For all κ, the κ-skeleta of the simplex Δ_{n-1} are shellable [20, p.286], hence $\mathcal{I}_{\Delta(n,\kappa)}$ is Cohen-Macaulay [16, Thm. 13.45]. The Eagon-Reiner theorem [4] now implies that $\mathcal{I}_{\Delta(n,n-\kappa)}$ has a linear minimal free resolution. Applying Alexander duality shows that $\mathcal{I}_{\Delta(n,k)}$ also has a linear minimal free resolution and is Cohen-Macaulay. □

If the minimal free resolution of an ideal $\mathcal{I}$ has the j-th term F_j, the *graded Betti numbers* are defined to be $b_{j,u} := \dim F_{j,u}$.

Proposition 3.6. *The graded Betti numbers of $\mathcal{I}_{\Delta(n,\kappa)}$ are, writing $F_{j,q}$ for the degree q term in the j-th term in the minimal free resolution of $\mathcal{I}_{\Delta(n,\kappa)}$,*

$$\dim F_{j,j+\kappa} = \binom{n}{\kappa+j}\binom{\kappa-1+j}{j},$$

and the graded Betti numbers are zero in all degrees other than $j+\kappa$.

Proof. By Lemma 3.5, the minimal free resolution of $Sym(V)/\mathcal{I}_{\Delta(n,\kappa)}$ is linear. Hence, there can be no cancellation in the Hilbert series, and the dimensions of the graded Betti numbers may be read off from the numerator of the Hilbert series. As the numerator of the Hilbert series is the h-vector of the $\kappa-2$ skeleton of Δ_{n-1} (by [16, Cor 1.15] and remarks following it), the result follows. □

Example 3.7. Consider the ideal $\mathcal{I}_{\Delta(5,3)}$, consisting of the ten square-free cubic monomials in five variables. The graded Betti numbers $b_{i,j}$ are displayed in a *Betti table*; starting at position $(0,0)$, the entry (reading right and down) in position (i,j) is $b_{i,i+j}$. The shift in the second index allows the Betti table to reflect the regularity of M, that is, the largest j such the Betti table is non-zero in row j. The Betti table of the minimal free resolution of $\mathcal{I}_{\Delta(5,3)}$ is

total	10	15	6
0	–	–	–
1	–	–	–
2	–	–	–
3	10	15	6

So, for example, $\dim F_{1,4}(\mathcal{I}_{\Delta(5,3)}) = 15$.

3.2. **The minimal free resolution from a representation-theoretic perspective.** First, to fix notation, the ideal is generated in degree κ by $x_{i_1}\cdots x_{i_\kappa}$, with $I\subset[n]$ and $|I|=\kappa$. Introduce the notation $\tilde{\mathfrak{S}}_\kappa = \mathfrak{S}_\kappa\times\mathfrak{S}_{n-\kappa}\subset\mathfrak{S}_n$, and if π is a partition of κ, write $\widetilde{[\pi]} = [\pi]\times[n-\kappa]$ for the $\tilde{\mathfrak{S}}_\kappa$-module that is $[\pi]$ as an $\mathfrak{S}_\kappa$-module and trivial as an $\mathfrak{S}_{n-\kappa}$-module. Recall that for finite groups $H\subset G$, and an H-module W, $Ind_H^G W = \mathbb{C}[G]\otimes_{\mathbb{C}[H]}W$ is the induced G-module, which in particular has dimension equal to $(\dim W)|G|/|H|$, and that $\dim[\pi]$ is given by the hook-length formula. These two facts give the dimensions asserted below.

As an $\mathfrak{S}_n$-module the space of generators is

$$M_1 = Ind_{\tilde{\mathfrak{S}}_\kappa}^{\mathfrak{S}_n}\widetilde{[\kappa]} = \bigoplus_{j=0}^{\min\{\kappa,n-\kappa\}}[n-j,j],$$

and it has dimension $\binom{n}{\kappa}$.

Proposition 3.8. *The generator of the j-th term in the minimal free resolution of $\mathcal{I}^{x_1\cdots x_n,\kappa}$, as an $\mathfrak{S}_n$-module, is*

$$M_{j,\kappa+j-1} = Ind_{\tilde{\mathfrak{S}}_{\kappa+j-1}}^{\mathfrak{S}_n}\widetilde{[\kappa,1^{j-1}]}, \tag{10}$$

which has dimension $\binom{\kappa+j-1}{j}\binom{n}{\kappa+j}$.

Proof. Let $I\subset[n]$ have cardinality $\kappa-1$, and let $i,j\in[n]\backslash I$ be distinct. Then M_2 has generators $S_{I,ij} := x_{i_1}\cdots x_{i_\kappa}x_i\otimes x_j - x_{i_1}\cdots x_{i_\kappa}x_j\otimes x_i$. That these map to zero and are linearly independent is clear, and since they span a space of the correct dimension, these must be the generators of M_2. The $\mathfrak{S}_{\kappa+1}$ action on $I\cup i\cup j$ is by $[\kappa,1]$.

In general, M_{j+1} has a basis

$$S_{I,u_1,\cdots,u_j} = \sum_\alpha (-1)^{\alpha+1} x_{i_1}\cdots x_{i_\kappa} x_{u_\alpha} \otimes x_{u_1} \wedge \cdots \wedge \hat{x}_{u_\alpha} \wedge \cdots \wedge x_{u_j}.$$

It is clear $S_{I,u_1,\cdots,u_j}$ is a syzygy, and for each fixed I it has the desired $\mathfrak{S}_{\kappa+j}$-action, and the number of such equals the dimension of M_j. □

4. On the minimal free resolution of the ideal generated by sub-permanents

Let $E, F = \mathbb{C}^n$, $V = E \otimes F$, and let $\mathcal{I}_\kappa^{\mathrm{perm}_n,\kappa} \subset S^\kappa(E\otimes F)$ denote the span of the sub-permanents of size κ and let $\mathcal{I}^{perm_\kappa} \subset Sym(E\otimes F)$ denote the ideal it generates. Note that $\dim \mathcal{I}_\kappa^{perm_\kappa} = \binom{n}{\kappa}^2$. Fix complete flags $0 \subset E_1 \subset \cdots \subset E_n = E$ and $0 \subset F_1 \subset \cdots \subset F_n = F$. Write $\mathfrak{S}_{E_j}$ for the copy of $\mathfrak{S}_j$ acting on E_j and similarly for F.

Write $T_E \subset SL(E)$ for the maximal torus (diagonal matrices). By [15], the subgroup G_{perm_n} of $GL(E\otimes F)$ preserving the permanent is $[(T_E \times \mathfrak{S}_E) \times (T_F \times \mathfrak{S}_F)] \ltimes \mathbb{Z}_2$, divided by the image of the n-th roots of unity.

As an $\mathfrak{S}_{E_n} \times \mathfrak{S}_{F_n}$-module the space $\mathcal{I}_\kappa^{\mathrm{perm}_n,\kappa}$ decomposes as

(11)
$$Ind_{\tilde{\mathfrak{S}}_{E_\kappa}\times\tilde{\mathfrak{S}}_{F_\kappa}}^{\mathfrak{S}_{E_n}\times\mathfrak{S}_{F_n}} \widetilde{[\kappa]}_{E_\kappa} \otimes \widetilde{[\kappa]}_{F_\kappa} = ([n]_E \oplus [n-1,1]_E \oplus \cdots \oplus [n-\kappa,\kappa]_E) \otimes ([n]_F \oplus [n-1,1]_F \oplus \cdots \oplus [n-\kappa,\kappa]_F).$$

4.1. The linear strand.

Example 4.1. The space of linear syzygies $M_{2,\kappa+1} := \ker(\mathcal{I}_\kappa^{\mathrm{perm}_n,\kappa} \otimes V \to S^{\kappa+1}V)$ is the $\mathfrak{S}_{E_n} \times \mathfrak{S}_{F_n}$-module

$$M_{2,\kappa+1} = Ind_{\tilde{\mathfrak{S}}_{E_{\kappa+1}}\times\tilde{\mathfrak{S}}_{F_{\kappa+1}}}^{\mathfrak{S}_{E_n}\times\mathfrak{S}_{F_n}} (\widetilde{[\kappa+1]}_{E_{\kappa+1}} \otimes \widetilde{[\kappa,1]}_{F_{\kappa+1}} \oplus \widetilde{[\kappa,1]}_{E_{\kappa+1}} \otimes \widetilde{[\kappa+1]}_{F_{\kappa+1}}).$$

This module has dimension $2\kappa\binom{n}{\kappa+1}^2$. A spanning set for it may be obtained geometrically as follows: for each size $\kappa+1$ sub-matrix, perform the permanental "tensor Laplace expansion" along a row or column, then perform a second tensor Laplace expansion about a row or column and take the difference. An independent set of such for a given size $\kappa+1$ sub-matrix may be obtained from the expansions along the first row minus the expansion along the j-th for $j = 2, \cdots, \kappa+1$, and then from the expansion along the first column minus the expansion along the j-th, for $j = 2, \cdots, \kappa+1$.

Remark 4.2. Compare this with the space of linear syzygies for the determinant, which has dimension $\frac{2\kappa(n+1)}{n-\kappa}\binom{n}{\kappa+1}^2$. The ratio of their sizes is $\frac{n+1}{n-\kappa}$, so, e.g., when $\kappa \sim \frac{n}{2}$, the determinant has about twice as many linear syzygies, and if κ is close to n, one gets nearly n times as many.

Theorem 4.3. $\dim M_{j+1,\kappa+j} = \binom{n}{\kappa+j}^2\binom{2(\kappa+j-1)}{j}$. *As an $\mathfrak{S}_n \times \mathfrak{S}_n$-module,*

(12)
$$M_{j+1,\kappa+j} = Ind_{\tilde{\mathfrak{S}}_{E_{\kappa+j}}\times\tilde{\mathfrak{S}}_{F_{\kappa+j}}}^{\mathfrak{S}_{E_n}\times\mathfrak{S}_{F_n}} \Big(\bigoplus_{a+b=j} [\widetilde{\kappa+b,1^a}]_{E_{\kappa+j}} \otimes [\widetilde{\kappa+a,1^b}]_{F_{\kappa+j}}\Big).$$

The $\binom{n}{\kappa+j}^2$ is just the choice of a size $\kappa+j$ submatrix, the $\binom{2(\kappa+j-1)}{j}$ comes from choosing a set of j elements from the set of rows union columns. Naïvely there are $\binom{2(\kappa+j)}{j}$ choices but there is redundancy as with the choices in the description of M_2.

Proof. The proof proceeds in two steps. We first get "for free" the minimal free resolution of the ideal generated by $S^\kappa E \otimes S^\kappa F$. Write the generating modules of this resolution as $\tilde{M}_j$. We then locate the generators of the linear strand of the minimal free resolution of our ideal, whose generators we denote $M_{j+1,\kappa+j}$, inside $\tilde{M}_{j+1,\kappa+j}$ and prove the assertion.

To obtain $\tilde{M}_{j+1}$, we use the involution ω on the space of symmetric functions (see, e.g. [14, §I.2]) that takes the Schur function s_π to $s_{\pi'}$. This involution extends to an endofunctor of $GL(V)$-modules and hence of $GL(E)\times GL(F)$-modules, taking $S_\lambda E\otimes S_\mu F$ to $S_{\lambda'}E\otimes S_{\mu'}F$ (see [1, §2.4]). This is only true as long as the dimensions of the vector spaces are sufficiently large, so to properly define it one passes to countably infinite dimensional vector spaces.

Applying this functor to the resolution (3), one obtains the resolution of the ideal generated by $S^\kappa E\otimes S^\kappa F\subset S^\kappa(E\otimes F)$. The $GL(E)\times GL(F)$-modules generating the linear component of the j-th term in this resolution are:

$$\begin{aligned}\tilde{M}_{j,j+\kappa-1} &= \bigoplus_{a+b=j-1} S_{(a,1^{\kappa+b})'}E\otimes S_{(b,1^{\kappa+a})'}F \\ &= \bigoplus_{a+b=j-1} S_{(\kappa+b+1,1^{a-1})}E\otimes S_{(\kappa+a+1,1^{b-1})}F.\end{aligned} \tag{13}$$

Moreover, by Corollary 2.5 and functoriality, the map from $S_{(\kappa+b+1,1^{a-1})}E\otimes S_{(\kappa+a+1,1^{b-1})}F$ into $\tilde{M}_{j-1,j+\kappa-1}$ is non-zero to the copies of $S_{(\kappa+b+1,1^{a-1})}E\otimes S_{(\kappa+a+1,1^{b-1})}F$ in

$$(S_{\kappa+b,1^{a-1}}E\otimes S_{\kappa+a+1,1^{b-2}F})\cdot(E\otimes F)\ \text{ and } (S_{\kappa+b+1,1^{a-2}}E\otimes S_{\kappa+a,1^{b-1}F})\cdot(E\otimes F),$$

when $a,b>0$.

Inside $S^\kappa E\otimes S^\kappa F$ is the ideal generated by the sub-permanents (11) which consists of the weight spaces $(p_1,\cdots,p_n)\times(q_1,\cdots,q_n)$, where all p_i,q_j are either zero or one. (Each sub-permanent has such a weight, and, given such a weight, there is a unique sub-permanent to which it corresponds.) Call such a weight space *regular*. Note that the set of regular vectors in any $E^{\otimes m}\otimes F^{\otimes m}$ (where $m\le n$ to have any) spans a $\mathfrak{S}_E\times\mathfrak{S}_F$-submodule.

The linear strand of the j-the term in the minimal free resolution of the ideal generated by (11) is thus a $\mathfrak{S}_E\times\mathfrak{S}_F$-submodule of $\tilde{M}_{j,j+\kappa-1}$. We claim this sub-module is the span of the regular vectors. In other words:

Lemma 4.4. $M_{j+1,\kappa+j}=(\tilde{M}_{j+1,\kappa+j})_{reg}$.

Assuming Lemma 4.4, Theorem 4.3 follows because if π is a partition of $\kappa+j$ then the weight $(1,\cdots,1)$ subspace of $S_\pi E_{\kappa+j}$, considered as an $\mathfrak{S}_{E_{\kappa+j}}$-module, is $[\pi]$ (see, e.g., [7]), and the space of regular vectors in $S_\pi E\otimes S_\mu F$ is $Ind_{\tilde{\mathfrak{S}}_{E_{\kappa+j}}\times\tilde{\mathfrak{S}}_{F_{\kappa+j}}}^{\mathfrak{S}_E\times\mathfrak{S}_F}\widetilde{[\pi]}_E\otimes\widetilde{[\mu]}_F$. □

Before proving Lemma 4.4 we establish conventions for the inclusions $S_{q+1,1^p}E\subset S_{q+1,1^{p-1}}E\otimes E$ and $S_{q+1,1^p}E\subset S_{q,1^p}E\otimes E$.

Let $\Theta(p,q): S_{q+1,1^p}E\to S_{q+1,1^{p-1}}E\otimes E$ be the $GL(E)$-module map defined such that the following diagram commutes:

$$\begin{array}{ccc} S^qE\otimes\Lambda^{p+1}E & \to & S_{q+1,1^p}E \\ \downarrow & & \downarrow\Theta(p,q) \\ S^qE\otimes E\otimes\Lambda^pE & \to & S_{q+1,1^{p-1}}E\otimes E\end{array},$$

where the left vertical map is the identity tensored with the polarization $\Lambda^{p+1}E\to\Lambda^pE\otimes E$.

We define two $GL(E)$-module maps $S^qE\otimes\Lambda^{p+1}E\to S^{q-1}E\otimes E\otimes\Lambda^{p+1}E$: σ_1, which is the identity on the second component and polarization on the first, i.e. $S^qE\to S^{q-1}E\otimes E$, and σ_2, which is defined to be the composition of

$$S^qE\otimes\Lambda^{p+1}E\to(S^{q-1}E\otimes E)\otimes(\Lambda^pE\otimes E)\to(S^{q-1}E\otimes E)\otimes(\Lambda^pE\otimes E)\to S^{q-1}E\otimes E\otimes\Lambda^{p+1}E$$

where the first map is two polarizations, the second map swaps the two copies of E and the last is the identity times skew-symmetrization. Let $\Sigma(p,q): S_{q+1,1^p}E\to S_{q+1,1^{p-1}}E\otimes E$ denote the

unique (up to scale) $GL(E)$-module inclusion (unique because $S_{q+1,1^p}E$ has multiplicity one in $S_{q+1,1^{p-1}}E\otimes E$). A short calculation shows that the following diagram is commutative:

$$\begin{array}{ccc} S^qE\otimes\Lambda^{p+1}E & \to & S_{q+1,1^p}E \\ \sigma_2 - p\sigma_1 \downarrow & & \downarrow \Sigma(p,q) \\ S^{q-1}E\otimes E\otimes\Lambda^{p+1}E & \to & S_{q,1^p}E\otimes E. \end{array}$$

Proof of Lemma 4.4. We work by induction, the case $j=1$ was discussed above. Assume the result has been proven up to $M_{j,\kappa+j-1}$ and consider $M_{j+1,\kappa+j}$. It must be contained in $M_{j,\kappa+j-1}\otimes(E\otimes F)$, so all its weights are either regular, or such that one of the p_i's is 2, and/or one of the q_i's is 2, and all other p_u, q_u are zero or 1. Call such a weight *sub-regular*. It remains to show that no linear syzygy with a sub-regular weight can appear. To do this we show that no sub-regular weight vector in $(M_{j,\kappa+j})_{subreg}$ maps to zero in $(M_{j-1,\kappa+j-1})_{reg}\cdot(E\otimes F)$.

First consider the case where both the E and F weights are sub-regular, then (because the space is a $\mathfrak{S}_E\times\mathfrak{S}_F$-module), the weight $(2,1,\cdots,1,0,\cdots,0)\times(2,1,\cdots,1,0,\cdots,0)$ must appear in the syzygy. But the only way for this to appear is to have a term of the form $T\cdot x_1^1$, which cannot map to zero because, since x_1^1 is a non-zero-divisor in $Sym(V)$, our syzygy is a syzygy of degree zero multiplied by x_1^1. But by minimality no such syzygy exists.

Finally consider the case where there is a vector of weight $(2,1^{j+\kappa-2})\times(1^{j+\kappa})$ appearing. Consider the set of vectors of this weight as a module for $\mathfrak{S}_{j+\kappa-2}\times\mathfrak{S}_{j+\kappa}$. This module is

$$\bigoplus_{a+b=j}[\kappa+a,1^b]/[2]\otimes[\kappa+b,1^a]. \tag{14}$$

Here

$$[\kappa+a,1^b]/[2] = [\kappa+a-2,1^b]\oplus[\kappa+a-1,1^{b-1}]$$

is called a *skew Specht module*.

By Howe-Young duality and Corollary 2.5 if $a,b>0$, $S_{\kappa+a,1^b}E\otimes S_{\kappa+b,1^a}F\subset M_{a+b+1,\kappa+a+b}$ maps non-zero to the two distinguished copies of the same module in $M_{a+b,\kappa+a+b}$. This in turn implies that the two distinguished copies of $S_{\kappa+a,1^b}E\otimes S_{\kappa+b,1^a}F\subset M_{a+b,\kappa+a+b}$, each map non-zero to $M_{a+b-1,\kappa+a+b}$.

The module (14) will take image inside

$$\bigoplus_{c+d=j-1} Ind^{\mathfrak{S}_{j+\kappa-1}\times\mathfrak{S}_{j+\kappa+1}}_{(\mathfrak{S}_{j+\kappa-2}\times\mathfrak{S}_1)\times(\mathfrak{S}_{j+\kappa}\times\mathfrak{S}_1)}([\kappa+c,1^d]/[2]\otimes[1])\otimes([\kappa+d,1^c]\otimes[1]).$$

Fix a term $[\kappa+b,1^a]$ on the right hand side and examine the map on the left hand side. It is a map

$$[\kappa+a,1^b]/[2]\to Ind^{\mathfrak{S}_{j+\kappa-1}}_{\mathfrak{S}_{j+\kappa-2}\times\mathfrak{S}_1}([\kappa+a,1^{b-1}]/[2]\otimes[1])\oplus Ind^{\mathfrak{S}_{j+\kappa-1}}_{\mathfrak{S}_{j+\kappa-2}\times\mathfrak{S}_1}([\kappa+a-1,1^b]/[2]\otimes[1]).$$

If $b>0$, the map to the first summand is the restriction of the map $\Theta(b,\kappa+a): S_{\kappa+a+1,1^b}E\to S_{\kappa+a+1,1^{b-1}}E\otimes E$, and, due to the fact that it has to map to a sub-regular weight, there is no polarization because the basis vector e_1 has to stay on the left hand side. So the map is the identity, thus injective.

It remains to show that for $b=0$, the map corresponding to the summand $b=0, a=j$ which is the restriction of the injective map $\Sigma(0,\kappa+j-2): S_{\kappa+j-1}E\to S_{\kappa+j-2}E\otimes E$ tensored with the map $\Theta(j-1,\kappa)$ injects into the cokernel of the summand corresponding to $c=0, d=j-1$ modulo the image of the map coming from the summand $a=1, b=j-1$. Both modules consist of just two irreducible $\mathfrak{S}_{E_{j+\kappa-1}}\times\mathfrak{S}_{F_{j+\kappa-1}}$-modules and, using formulas for Σ and Θ, the map is injective. This concludes the proof. □

Example 4.5. For small n and κ, computer computations show no additional first syzygies on the $\kappa \times \kappa$ sub-permanents of a generic $n \times n$ matrix (besides the linear syzygies) in degree less than the Koszul degree 2κ. For example, for $\kappa = 3$ and $n = 5$, there are 100 cubic generators for the ideal and 5200 minimal first syzygies of degree six. There can be at most $\binom{100}{2} = 4950$ Koszul syzygies, so there must be additional non-Koszul first syzygies.

4.2. **The Hilbert function in the case $\kappa = 2$.** First, the Hilbert polynomial:

Theorem 4.6. *For the ideal $\mathcal{I}^{\mathrm{perm}_n,2}$ of 2×2 permanents of an $n \times n$ matrix, the Hilbert polynomial of $Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}$ is*

$$\sum_{i=0}^{n} f_i \binom{t-1}{i}, \tag{15}$$

where f_i is the i^{th} entry in the vector

$$\left[n^2, \binom{n^2}{2} - \binom{n}{2}^2, 2\binom{n}{2}^2 + 2n\binom{n}{3}, 2n\binom{n}{4}, 2n\binom{n}{5}, \dots, 2n\binom{n}{n}\right]$$

Proof. [13, Thm. 3.2] gives a Gröbner basis for $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$, the radical of $\mathcal{I}^{\mathrm{perm}_n,2}$, and by [13, Thm. 3.3], $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2}$ has finite length, so vanishes in high degree. The Hilbert polynomial only measures dimension asymptotically, so

$$HP(Sym(V)/\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}, t) = HP(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}, t).$$

By [13], for any diagonal term order, the Gröbner basis for $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$ is given by quadrics of the form

$$x_{ij}x_{kl} + x_{kj}x_{il} \text{ with } i < k,\ j < l,$$

and five sets of cubic monomials

$$\begin{array}{lcc} x_{i_1j_1}x_{i_1j_2}x_{i_2j_3} & i_1 > i_2 & j_1 < j_2 < j_3 \\ x_{i_1j_1}x_{i_2j_2}x_{i_2j_3} & i_1 > i_2 & j_1 < j_2 < j_3 \\ x_{i_1j_1}x_{i_2j_1}x_{i_3j_2} & i_1 < i_2 < i_3 & j_1 > j_2 \\ x_{i_1j_1}x_{i_2j_2}x_{i_3j_2} & i_1 < i_2 < i_3 & j_1 > j_2 \\ x_{i_1j_1}x_{i_2j_2}x_{i_3j_3} & i_1 < i_2 < i_3 & j_1 > j_2 > j_3. \end{array} \tag{16}$$

The key observation is that all the cubic monomials are square-free, as are the initial terms of the quadrics. Thus the initial ideal of $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}$ is a square-free monomial ideal and corresponds to the Stanley-Reisner ideal of a simplicial complex Δ. By [18, Lem. 5.2.5], the Hilbert polynomial is as in Equation (15), where f_i is the number of i-dimensional faces of Δ. As the vertex set of Δ corresponds to all lattice points (i, j) with $1 \le i, j \le n$, it is immediate that $f_0 = n^2$.

Since $x_{ij}x_{kl}$ is a non-face if $i < k$, $j < l$, no edge connects a southwest lattice point to a northeast lattice point. Hence, the edges of Δ consist of all pairs $(i, j), (k, l)$ with $i \ge k$ and $j \ge l$, of which there are $\binom{n^2}{2} - \binom{n}{2}^2$.

Next, consider the triangles of Δ. Equation (16) says there are no triangles in Δ of the types in Figure 2. Also, there are no triangles which contain an edge connecting vertices at positions (i, j) and (k, l) with $i < k$,$j < l$. Thus, the only triangles in Δ are right triangles, but with hypotenuse sloping from northwest to southeast. For a lattice point v at position (d, e) there are exactly $(d-1)(e-1)$ right triangles having v as their unique north-most vertex. In the rightmost column n, there are no such triangles, in the next to last column $n-1$ there are $(n-1) + (n-2) + \cdots = \binom{n}{2}$ such triangles. Continuing this way yields a total count of

$$(n-1)\binom{n}{2} + (n-2)\binom{n}{2} + \cdots 2\binom{n}{2} + \binom{n}{2} = \binom{n}{2}^2$$

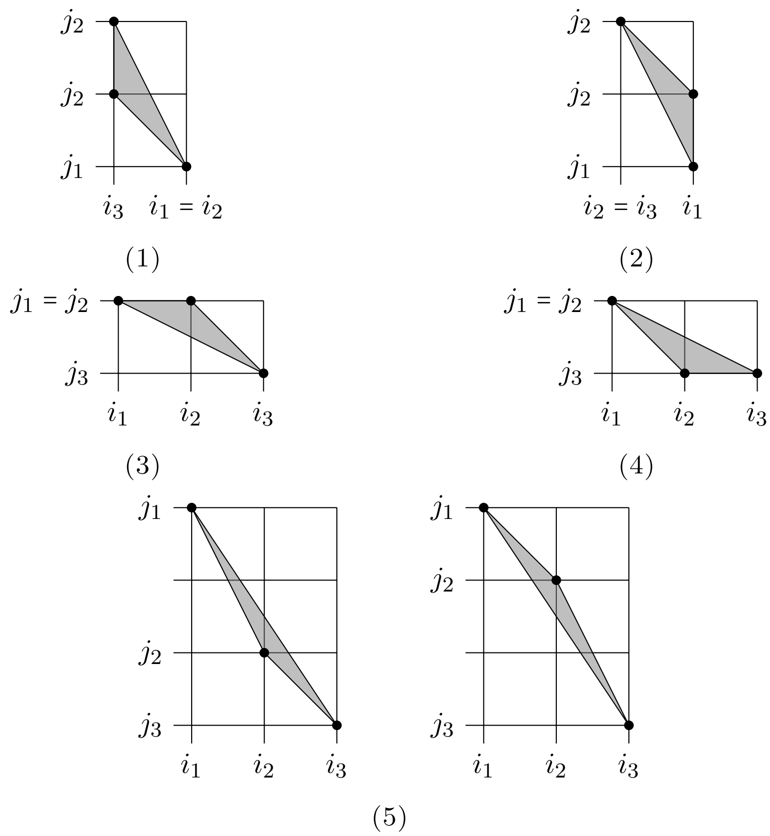

FIGURE 2. Non-triangles of Δ from Equation (16)

such right triangles, and taking into account the right triangles for which v is the unique south-most vertex doubles this number.

However, this count neglects thin triangles–those which have all vertices in the same row or column. Since the number of thin triangles is $2n\binom{n}{3}$, the final count for the triangles of Δ is

$$2\binom{n}{2}^2 + 2n\binom{n}{3}.$$

For tetrahedra, the conditions of Equation (16) imply that there can only be thin tetrahedra, and an easy count gives $2n\binom{n}{4}$ such. The same holds for higher dimensional simplices, and concludes the proof. $\square$

Corollary 4.7. *For the ideal $\mathcal{I}^{\mathrm{perm}_n,2}$ of 2×2 permanents of an $n\times n$ matrix, the Hilbert function of $Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}$ is, when $3 \le t \le n$,*

$$HF(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}, t) = \binom{n}{t}^2 + HP(Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2}, t), \tag{17}$$

and it equals the Hilbert polynomial for $t > n$.

Proof. The Hilbert function of $\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2}$ in degree t is $\binom{n}{t}^2$ by [13, Thm. 3.3]. The result follows by combining Theorem 15 with the short exact sequence

$$0 \longrightarrow \sqrt{\mathcal{I}^{\mathrm{perm}_n,2}}/\mathcal{I}^{\mathrm{perm}_n,2} \longrightarrow Sym(V)/\mathcal{I}^{\mathrm{perm}_n,2} \longrightarrow Sym(V)/\sqrt{\mathcal{I}^{\mathrm{perm}_n,2}} \longrightarrow 0,$$

and additivity of the Hilbert function. □

For the purposes of comparing with other ideals, we rephrase this as:

Theorem 4.8. $\dim \mathcal{I}_2^{\mathrm{perm}_n,2} = \binom{n}{2}^2$. *For $3 \le t \le n$:*

$$\dim \mathcal{I}_t^{perm_n,2} = \binom{n^2+t-1}{t} - [\binom{n}{t}^2 + n^2 + (t-1)(\binom{n^2}{2} - \binom{n}{2}^2) + 2\binom{t-1}{2}(\binom{n}{2}^2 + n\binom{n}{3}) + 2n\sum_{j=3}^{t-1}\binom{t-1}{j}\binom{n}{j+1}]$$

and for $t > n$:

$$\dim \mathcal{I}_t^{perm_n,2} = \binom{n^2+t-1}{t} - [n^2 + (t-1)(\binom{n^2}{2} - \binom{n}{2}^2) + 2\binom{t-1}{2}(\binom{n}{2}^2 + n\binom{n}{3}) + 2n\sum_{j=3}^{t-1}\binom{t-1}{j}\binom{n}{j+1}]$$

and the latter formula is $\dim S^t\mathbb{C}^{n^2}$ minus the Hilbert polynomial for all t.

References

1. Kaan Akin and Jerzy Weyman, *Primary ideals associated to the linear strands of Lascoux's resolution and syzygies of the corresponding irreducible representations of the Lie superalgebra* $\mathbf{gl}(m|n)$, J. Algebra **310** (2007), no. 2, 461–490. MR 2308168 (2009c:17007)
2. N. Alon and M. Tarsi, *A nowhere-zero point in linear mappings*, Combinatorica **9** (1989), no. 4, 393–395. MR 1054015 (92a:11147)
3. Riccardo Biagioli, Sara Faridi, and Mercedes Rosas, *Resolutions of De Concini-Procesi ideals of hooks*, Comm. Algebra **35** (2007), no. 12, 3875–3891. MR 2371263 (2008i:13018)
4. John A. Eagon and Victor Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), no. 3, 265–275. MR 1633767 (99h:13017)
5. Klim Efremenko, J.M. Landsberg, Hal Schenck, and Jerzy Weyman, *On the method of shifted partial derivatives in complexity theory*, preprint (2016).
6. William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249 (93a:20069)
7. David A. Gay, *Characters of the Weyl group of $SU(n)$ on zero weight spaces and centralizers of permutation representations*, Rocky Mountain J. Math. **6** (1976), no. 3, 449–455. MR MR0414794 (54 #2886)
8. Ankit Gupta, Pritish Kamath, Neeraj Kayal, and Ramprasad Saptharishi, *Approaching the chasm at depth four*, Proceedings of the Conference on Computational Complexity (CCC) (2013).
9. George A. Kirkup, *Minimal primes over permanental ideals*, Trans. Amer. Math. Soc. **360** (2008), no. 7, 3751–3770. MR 2386244 (2009a:13017)
10. J. M. Landsberg, *Geometric complexity theory: an introduction for geometers*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **61** (2015), no. 1, 65–117. MR 3343444
11. J. M. Landsberg and Giorgio Ottaviani, *Equations for secant varieties of Veronese and other varieties*, Ann. Mat. Pura Appl. (4) **192** (2013), no. 4, 569–606. MR 3081636
12. Alain Lascoux, *Syzygies des variétés déterminantales*, Adv. in Math. **30** (1978), no. 3, 202–237. MR 520233 (80j:14043)
13. Reinhard C. Laubenbacher and Irena Swanson, *Permanental ideals*, J. Symbolic Comput. **30** (2000), no. 2, 195–205. MR 1777172 (2001i:13039)
14. I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 1354144 (96h:05207)
15. Marvin Marcus and F. C. May, *The permanent function*, Canad. J. Math. **14** (1962), 177–189. MR MR0137729 (25 #1178)
16. Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR 2110098 (2006d:13001)

17. P. Roberts, *A minimal free complex associated to the minors of a matrix*, unpublished (1978).
18. Hal Schenck, *Computational algebraic geometry*, London Mathematical Society Student Texts, vol. 58, Cambridge University Press, Cambridge, 2003. MR 2011360 (2004k:13001)
19. Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003. MR MR1988690 (2004d:13020)
20. Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR 1311028 (96a:52011)

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